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On the loss of power of a test based on simulation

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Abstract

In order to obtain a sufficiently accurate approximation of the distribution under the suitable conditions by the Monte Carlo simulation, the number of replications may be very large. But, it is shown in the paper that such a large sample is not necessary if one wants to get a randomized test of exact size which has a negligible loss of efficiency in comparison with the best non-randomized test in some sense.

1. Introduction

As sophisticated programs come to be more easily available in statistical analysis, simulation techniques are more often resorted to. Thus when exact test procedures are difficult to be calculated and usual asymptotic approximations are not exact enough, simulation technique is often applied (see, *e.g.* Johnson (1987), Rubinstein (1981), Rubinstein and Melamed (1998)). But it often happens when we want to approximate the distribution with sufficient accuracy under the hypothesis by Monte Carlo simulation, the repeated number required is very big.

The purpose of this paper is to show that it is not necessary to have such a big sample, if we have in mind that our object can be considered to obtain a randomized test of exact size with negligible loss of efficiency compared with the best non-randomized test, which goal can be achieved with relatively small Monte Carlo sample.

2. Loss of the power of the test

Consider the following situation, let $X_1, \ldots, X_n$ be random variables according to some joint distribution P_θ characterized by a real parameter θ.

Suppose that it is required to test the simple hypothesis $H : \theta = \theta_0$ against the alternative $\theta \neq \theta_0$ with level α. A test procedure based on the test statistic $T^* = t(X_1, \ldots, X_n)$ rejecting the hypothesis if $T^* > t_\alpha$, is shown to have optimum (in some or other sense) property.

However, it often happens that the exact critical point t_α is difficult to calculate, and the approximations (*e.g.* based on the asymptotic expansion) are not necessarily accurate. Then we have to resort to simulation. N replications of the set of n values $X'_{1i}, \ldots, X'_{ni}$ ($i = 1, \ldots, N$) which are independently distributed according to the same joint distribution with $X_1, \ldots, X_n$ with $\theta = \theta_0$. N values of statistic $T_i = t(X'_{1i}, \ldots, X'_{ni})$ are calculated and the hypothesis rejected if $T^* > T_{(m)}$, where $T_{(m)}$ is the m–th largest value in the order set of the values $T_1, \ldots, T_N$. Then, assuming that the distribution of T is continuous with zero probability for the ties, we have

$$P_{\theta_0}\left\{T^* > T_{(m)}\right\} = \frac{m}{N+1}.$$

Hence, if $m = (N+1)\alpha$, we have the test procedure of exact size. For the power of the test we define

$$Q_\theta(t) := P_\theta\{T^* > t\}.$$

Then $Q_{\theta_0}(t_\alpha) = \alpha$ and $\beta^*(\theta) := Q_\theta(t_\alpha)$ is the power of the "optimum" test at $\theta \neq \theta_0$. The power function of the above randomized test is given as

$$\beta(\theta) := E_\theta[P_\theta\{T^* > T_{(m)} | T_{(m)}\}] = E_\theta[Q_\theta(T_m)].$$

Now we can expand it as

$$\begin{aligned} Q_\theta(T_{(m)}) =& Q_\theta(t_\alpha) + \left\{\frac{\partial}{\partial t} Q_\theta(t_\alpha)\right\}(T_{(m)} - t_\alpha) \\ &+ \frac{1}{2}\frac{\partial^2}{\partial t^2} Q_\theta(t_\alpha)(T_{(m)} - t_\alpha)^2 + o((T_{(m)} - t_\alpha)^2). \end{aligned}$$

Denoting

$$b := E_\theta[T_{(m)} - t_\alpha],$$

and

$$v := V_\theta(T_{(m)} - t_\alpha) = E_\theta[(T_{(m)} - t_\alpha)^2] - b^2,$$

we have

$$\begin{aligned} \beta(\theta) = E[Q_\theta(T_{(m)})] = Q_\theta(t_\alpha) + \left\{\frac{\partial}{\partial t} Q_\theta(t_\alpha)\right\} b + \frac{1}{2}\left\{\frac{\partial^2}{\partial t^2} Q_\theta(t_\alpha)\right\}(v^2 + b^2) \\ + o(v^2 + b^2). \end{aligned}$$

When $\theta = \theta_0$, we have

$$\begin{aligned} \alpha &= E_{\theta_0}[Q_{\theta_0}(T_{(m)})] \\ &= Q_{\theta_0}(t_\alpha) + \left\{\frac{\partial}{\partial t} Q_{\theta_0}(t_\alpha)\right\} b + \frac{1}{2}\left\{\frac{\partial^2}{\partial t^2} Q_{\theta_0}(t_\alpha)\right\}(v^2 + b^2) + o(v^2 + b^2). \end{aligned}$$

Since $Q_{\theta_0}(t_\alpha) = \alpha$, it follows that

$$\left\{\frac{\partial}{\partial t} Q_{\theta_0}(t_\alpha)\right\} b + \frac{1}{2}\left\{\frac{\partial^2}{\partial t^2} Q_{\theta_0}(t_\alpha)\right\}(v^2 + b^2) = o(v^2 + b^2),$$

$$\beta(\theta) = \beta^*(\theta) + \left\{\frac{\partial}{\partial t} Q_\theta(t_\alpha)\right\} b + \frac{1}{2}\left\{\frac{\partial^2}{\partial t^2} Q_\theta(t_\alpha)\right\}(v^2+b^2) + o(v^2+b^2)$$
$$= \beta^*(\theta) + \frac{1}{2}(v^2+b^2)\left\{\frac{\partial^2}{\partial t^2} Q_\theta(t_\alpha) - \frac{\frac{\partial}{\partial t} Q_\theta(t_\alpha)\frac{\partial^2}{\partial t^2} Q_{\theta_0}(t_\alpha)}{\frac{\partial}{\partial t} Q_{\theta_0}(t_\alpha)}\right\} + o(v^2+b^2).$$

The second term $\{\cdots\}$ represents the loss of the power due to randomization.

Now let $U_1 < \cdots < U_N$ be the order statistic of size N from the uniform distribution on the interval $(0,1)$. Then we can express

$$T_{(m)} = Q_\theta^{-1}(U_{(m)})$$
$$= Q_\theta^{-1}(\alpha) + \left\{\frac{\partial}{\partial u} Q_\theta^{-1}(\alpha)\right\}(U_{(m)} - \alpha)$$
$$+ \frac{1}{2}\left\{\frac{\partial^2}{\partial u^2} Q_\theta^{-1}(\alpha)\right\}(U_{(m)} - \alpha)^2 + o((U_{(m)} - \alpha)^2),$$

from which we obtain

$$E_\theta(T_{(m)}) = t_\alpha + \frac{1}{2}\left\{\frac{\partial^2}{\partial u^2} Q_\theta^{-1}(\alpha)\right\}\frac{\alpha(1-\alpha)}{N+2} + o\left(\frac{1}{N}\right),$$

$$E_{\theta_0}[(T_{(m)} - t_\alpha)^2] = \left\{\frac{\partial}{\partial u} Q_{\theta_0}^{-1}(\alpha)\right\}^2 \frac{\alpha(1-\alpha)}{N+2} + o\left(\frac{1}{N}\right)$$
$$= \frac{1}{\left\{\frac{\partial}{\partial t} Q_{\theta_0}(t_\alpha)\right\}^2}\frac{\alpha(1-\alpha)}{N+2} + o\left(\frac{1}{N}\right).$$

Consequently we have

$$\beta^*(\theta) - \beta(\theta) = -\frac{\alpha(1-\alpha)}{2(N+2)\left\{\frac{\partial}{\partial t} Q_{\theta_0}(t_\alpha)\right\}}$$
$$\cdot \left[\frac{\frac{\partial^2}{\partial t^2} Q_\theta(t_\alpha)}{\frac{\partial}{\partial t} Q_{\theta_0}(t_\alpha)} - \frac{\left\{\frac{\partial}{\partial t} Q_\theta(t_\alpha)\right\}\left\{\frac{\partial^2}{\partial t^2} Q_{\theta_0}(t_\alpha)\right\}}{\left\{\frac{\partial}{\partial t} Q_{\theta_0}(t_\alpha)\right\}^2}\right] + o\left(\frac{1}{N}\right).$$

3. Normal case

When T^* is distributed according to the normal distribution with mean 0 and variance 1 under the hypothesis and mean $\theta(>0)$ and variance σ^2 under the alternative hypothesis, we obtain

$$Q_{\theta_0}(t) = 1 - \Phi(t),$$
$$Q_\theta(t) = 1 - \Phi\left(\frac{t-\theta}{\sigma}\right),$$

where

$$\Phi(t) = \int_{-\infty}^{t} \phi(u)du \quad \text{with} \quad \phi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2},$$

hence

$$\frac{\partial}{\partial t} Q_{\theta_0}(t) = -\phi(t), \quad \frac{\partial^2}{\partial t^2} Q_{\theta_0}(t) = t\phi(t),$$
$$\frac{\partial}{\partial t} Q_{\theta}(t) = -\frac{1}{\sigma}\phi\left(\frac{t-\theta}{\sigma}\right), \quad \frac{\partial^2}{\partial t^2} Q_{\theta}(t) = \frac{t-\theta}{\sigma^3}\phi\left(\frac{t-\theta}{\sigma}\right).$$

Therefore we have

$$\beta(\theta) - \beta^*(\theta) = \frac{\alpha(1-\alpha)}{2(N+2)} \cdot \frac{1}{t_\alpha \phi^2(t_\alpha)} \phi\left(\frac{t_\alpha - \theta}{\sigma}\right)\left(\frac{t}{\sigma} - \frac{t_\alpha - \theta}{\sigma^3}\right).$$

If $\sigma^2 = 1$, then

$$\beta(\theta) - \beta^*(\theta) = \frac{\alpha(1-\alpha)}{2(N+2)} \cdot \frac{\theta}{t_\alpha \phi^2(t_\alpha)} \phi(t_\alpha - \theta),$$

hence, for a large t_α and small θ,

$$\Delta(\theta) := \beta(\theta) - \beta^*(\theta) \approx \frac{\alpha(1-\alpha)}{2(N+2)} \cdot \frac{\theta}{\phi(t_\alpha)} \cdot \frac{1}{t_\alpha} \exp\left(-\frac{\theta^2}{2} + \theta t_\alpha\right). \tag{1}$$

From (1) we have

$$\begin{aligned}\frac{2t_\alpha \phi(t_\alpha)}{\alpha(\alpha-1)}(N+2)\Delta(\theta) &\approx \theta \exp\left(-\frac{\theta^2}{2} + \theta t_\alpha\right) \\ &= \theta \exp\left\{-\frac{1}{2}(\theta - t_\alpha)^2 + \frac{1}{2}t_\alpha^2\right\} \end{aligned} \tag{2}$$

for a large t_α and small θ. Then it follows that the value maximizing (2) is given by

$$\theta = \frac{1}{2}\left(t_\alpha + \sqrt{t_\alpha^2 + 4}\right). \tag{3}$$

If $\alpha = 0.05$, then $t_\alpha \fallingdotseq 1.64$, which yields $\theta \fallingdotseq 2.11$ from (3). From (2) we also have $N \fallingdotseq 100.55$ for $c = 0.01$. The values of N satisfying $\Delta(\theta) = c$ for $\sigma^2 = 1$ and $\alpha = 0.05$ are given below.

Table 1. The values of the solution N of $\Delta(\theta) = c$ for $\sigma^2 = 1$, $\alpha = 0.05$.

$c \diagdown \theta$	0.50	1.00	1.50	2.11	2.50	3.00
0.01	12.06	41.99	78.38	100.55	91.92	62.86
0.005	26.12	85.98	158.77	203.10	185.85	127.72
0.001	138.60	437.88	801.83	1023.48	937.24	646.60

References

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